

BELYI MAP FOR THE SPORADIC GROUP J_1

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ABSTRACT. We compute the genus 0 Belyi map for the sporadic Janko group J_1 of degree 266 and describe the applied method. This yields explicit polynomials having J_1 as a Galois group over $K(t)$, $[K : \mathbb{Q}] = 7$.

1. INTRODUCTION

Our main goal of this paper is to present a method to compute high degree genus 0 Belyi maps having prescribed monodromy. Using this technique we found explicit polynomials having the sporadic Higman-Sims group as Galois groups over $\mathbb{Q}(t)$, see [3], and more generally all Belyi maps of genus 0 having almost simple, primitive monodromy groups (not A_n , S_n) generated by rigid rational triples of degree between 50 and 250, see [4]. In the following we will apply this method to compute a Belyi map of degree 266 for the permutation triple $(x, y, z := (xy)^{-1}) \in S_{266}^3$, given in the ancillary data. It is of type

	x	y	$z = (xy)^{-1}$
cycle structure	7^{38}	$2^{128}.1^{10}$	$3^{87}.1^5$

and has the following properties:

- x, y generate the sporadic Janko group J_1 .
- (x, y, z) is of genus 0.
- The permutations x, y, z each lie in rational conjugacy classes.
- (x, y, z) has passport size 7.

Alternative techniques for computing Belyi maps can be found in [15], [14], [16], [12] and [7]. Recently, Monien [13] demonstrated another powerful method by computing a Belyi map for J_2 of degree 100.

2. METHOD OF COMPUTATION

Let $a := \text{ord}(x)$, $b := \text{ord}(y)$, $c := \text{ord}(z)$ and

$$\Delta := \left\langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_a \delta_b \delta_c = 1 \right\rangle.$$

Note that (x, y, z) is hyperbolic since $1/a + 1/b + 1/c < 1$. We now consider the embedding $\Delta \hookrightarrow \text{PSL}_2(\mathbb{R})$ described in [7, Proposition 2.5], where δ_a (resp. δ_b) is mapped to a hyperbolic rotation around i (resp. μi for some $\mu > 1$) of angle π/a (resp. π/b). Thus Δ acts on the upper half-plane \mathbb{H} via the natural action of $\text{PSL}_2(\mathbb{R})$ on \mathbb{H} and its fundamental domain is the hyperbolic kite with vertices $i, \gamma, \mu i, -\gamma$ for some $\gamma \in \mathbb{H}$. Furthermore, let φ

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denote the homomorphism from Δ onto $G := \langle x, y \rangle$ such that $\delta_a \mapsto x$ and $\delta_b \mapsto y$. With the notation $\Gamma := \varphi^{-1}(G_1) < \Delta$ we define

$$\Phi : \mathbb{H}/\Gamma \rightarrow \mathbb{H}/\Delta, z \bmod \Gamma \mapsto z \bmod \Delta.^1$$

This is a Belyi map, i.e. a three branch point covering, of degree d ramified over $i, \mu i$ and γ and its monodromy group is isomorphic to G , see for example [7] for more details.

One can now restrict Φ to a connected fundamental domain $D \subset \mathbb{H}$ by introducing an equivalence relation \sim on ∂D induced by the quotient structure of \mathbb{H}/Γ . After identifying \mathbb{H}/Δ with $\mathbb{P}^1(\mathbb{C})$ such that $i \mapsto 0$, $\mu i \mapsto 1$ and $\gamma \mapsto \infty$ it suffices to study the following induced Belyi map

$$F : D/\sim \rightarrow \mathbb{P}^1(\mathbb{C}).$$

In the following we describe how to get a Belyi map of type $\mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$, this is possible because our given permutation triple is of genus 0, thus $D/\sim \cong \mathbb{P}^1(\mathbb{C})$. The Schwarz-Christoffel Toolbox [6] for MATLAB [11] gives us an approximation of a conformal map

$$h_1 : D^\circ \rightarrow \mathbb{H}.$$

We can extend h_1 on \overline{D} such that $h_1(\partial D) = \partial \mathbb{H}$ and h_1 is well defined on D/\sim . Now \sim induces via h_1 a new equivalence relation \approx on $\partial \mathbb{H}$. In order to glue the corresponding edges on $\partial \mathbb{H}$ we apply the slit algorithm (see [10] and [2]). We therefore find a tree T in $\mathbb{P}^1(\mathbb{C})$ and a conformal map

$$h_2 : \mathbb{H} \rightarrow \mathbb{P}^1(\mathbb{C}) \setminus T$$

such that h_2 can be extended on $\overline{\mathbb{H}}$, $h_2(\partial \mathbb{H}) = T$ and h_2 is well defined on \mathbb{H}/\approx . This leads to the Belyi map

$$F \circ h_1^{-1} \circ h_2^{-1} : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$$

having the prescribed ramification data (x, y, z) over 0, 1 and ∞ .

Practically, the above method yields approximate preimages of 0, 1 and ∞ . Using Newton's method we compute this Belyi map with sufficiently high precision, allowing us to recognize its coefficients as algebraic numbers using the LLL algorithm [9] implementation provided by Magma [5].

3. COMPUTED RESULTS AND VERIFICATION

The computed Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q},$$

defined over the number field

$$K = \mathbb{Q}(\alpha) \text{ with } \alpha^7 - \alpha^6 - 2\alpha^4 - \alpha^3 + 2\alpha^2 + 2\alpha + 2 = 0$$

can be found in the ancillary file and its reduction modulo a prime ideal of norm 269 is given in the appendix.

With the Riemann-Hurwitz genus formula and the factorizations of p, q and r one can easily verify that $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$ is indeed a Belyi map, i.e. a three branch point covering, ramified over 0, 1 and ∞ .

¹For a topological space X and a group G acting on X let X/G denote the corresponding orbit space.

Verification of monodromy. Let $\mathfrak{p} = (5, 2 + \alpha)\mathcal{O}_K$ be the unique prime ideal in the ring of integers \mathcal{O}_K of K with norm 5. Then all coefficients of $p(X) - tq(X) \in K(t)[X]$ lie in the localization $R := \mathcal{O}_K[t]_{\mathfrak{p}\mathcal{O}_K[t]}$ of $\mathcal{O}_K[t]$ at the prime ideal $\mathfrak{p}\mathcal{O}_K[t]$. Denote by \bar{p} and \bar{q} the images under the canonical homomorphism $R[X] \rightarrow (R/\mathfrak{p}R)[X] \cong \mathbb{F}_5(t)[X]$. The decomposition algorithm, found in [1], implemented in Magma [5], yields that $\bar{p}/\bar{q} \in \mathbb{F}_5(X)$ is indecomposable, thus $A_{\mathbb{F}_5} := \text{Gal}(\bar{p}(X) - t\bar{q}(X) \in \mathbb{F}_5(t)[X])$ is primitive by Ritt's Theorem. Since $K(t)$ is the quotient field of R we find $A_{\mathbb{F}_5} \leq A_K := \text{Gal}(p(X) - tq(X) \in K(t)[X])$ using Dedekind reduction, see [8, VII, Theorem 2.9]. It follows that A_K is primitive.

Furthermore, note that $(p(t)q(X) - q(t)p(X))/(X - t) \in K(t)[X]$ is reducible², thus A_K is not 2-transitive. Since J_1 , A_{266} , and S_{266} are the only primitive groups of degree 266 we find $A_K = J_1$. It is well-known that the geometric monodromy group $G := \text{Gal}(p(X) - tq(X) \in \mathbb{C}(t)[X])$ is normal in A_K . Since J_1 is simple, we have $G = A_K = J_1$. \square

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APPENDIX: COMPUTED DATA

In this section we will present the computed Belyi map

$$f = \frac{p}{q} = 1 + \frac{r}{q}$$

in the following way: We take the prime ideal $\mathfrak{p} = (269, 207 + \alpha)\mathcal{O}_K$ of norm 269 and reduce the coefficients of $p, q \in K[X]$ by \mathfrak{p} to obtain polynomials \bar{p} and \bar{q} lying in $(\mathcal{O}_K/\mathfrak{p})[X] \cong \mathbb{F}_{269}[X]$. The results are

$$\begin{aligned} \bar{p}(X) &= (X + 224)^7. \\ &(X^2 + 201X + 17)^7. \\ &(X^2 + 225X + 175)^7. \\ &(X^3 + 146X + 146)^7. \\ &(X^3 + 7X^2 + 88X + 26)^7. \\ &(X^3 + 176X^2 + 124X + 227)^7. \\ &(X^6 + 77X^5 + 257X^4 + 141X^3 + 224X^2 + 252X + 140)^7. \\ &(X^6 + 105X^5 + 148X^4 + 149X^3 + 231X^2 + 132X + 250)^7. \\ &(X^6 + 116X^5 + 61X^4 + 230X^3 + 119X^2 + 114X + 201)^7. \\ &(X^6 + 257X^5 + 46X^4 + 100X^3 + 79X^2 + 188X + 127)^7 \end{aligned}$$

²The ancillary data contains a factor of $p(t)q(X) - q(t)p(X)$ of degree 11. This divisor was computed by using sufficiently enough specializations in t and factorizing the resulting polynomials giving rise to an interpolation.

and

$$\begin{aligned}
\bar{q}(X) = & (X^2 + 182X + 28)^3 \cdot \\
& (X^5 + 107X^4 + 235X^3 + 37X^2 + 91X + 15) \cdot \\
& (X^5 + 145X^4 + 46X^3 + 241X^2 + 163X + 209)^3 \cdot \\
& (X^{10} + 29X^9 + 90X^8 + 199X^7 + 20X^6 + 220X^5 \\
& + 18X^4 + 104X^3 + 212X^2 + 168X + 126)^3 \cdot \\
& (X^{10} + 41X^9 + 63X^8 + 219X^7 + 177X^6 + 124X^5 \\
& + 134X^4 + 246X^3 + 206X^2 + 41X + 18)^3 \cdot \\
& (X^{10} + 55X^9 + 247X^8 + 253X^7 + 161X^6 + 49X^5 \\
& + 242X^4 + 113X^3 + 235X^2 + 212X + 169)^3 \cdot \\
& (X^{10} + 67X^9 + 51X^8 + 12X^7 + 135X^6 + 116X^5 \\
& + 172X^4 + 265X^3 + 239X^2 + 45X + 119)^3 \cdot \\
& (X^{10} + 76X^9 + 193X^8 + 151X^7 + 120X^6 + 172X^5 \\
& + 192X^4 + 256X^3 + 29X^2 + 68X + 24)^3 \cdot \\
& (X^{10} + 98X^9 + 229X^8 + 244X^7 + 142X^6 + 180X^5 \\
& + 65X^4 + 188X^2 + 23X + 227)^3 \cdot \\
& (X^{10} + 164X^9 + 124X^8 + 10X^7 + 268X^6 + 169X^5 \\
& + 204X^4 + 222X^3 + 106X^2 + 30X + 200)^3 \cdot \\
& (X^{10} + 194X^9 + 187X^8 + 100X^7 + 204X^6 + 145X^5 \\
& + 224X^4 + 67X^3 + 105X^2 + 203X + 5)^3.
\end{aligned}$$

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